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A SINGULAR LIMIT PROBLEM FOR THE ROSENAU-KORTEWEG-DE VRIES -REGULARIZED LONG WAVE AND ROSENAU-KORTEWEG-DE VRIES EQUATION.

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ABSTRACT. We consider the Rosenau-Korteweg-de Vries-regularized long wave and Rosenau-Korteweg-de Vries equations, which contain nonlinear dispersive effects. We prove that, as the diffusion parameter tends to zero, the solutions of the dispersive equations converge to the unique entropy solution of a scalar conservation law. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the L^p setting.

1. Introduction

The dynamics of shallow water waves that is observed along lake shores and beaches has been a research area for the past few decades in the area of oceanography (see [1, 21]). There are several models proposed in this context: Korteweg-de Vries (KdV) equation, Boussinesq equation, Peregrine equation, regularized long wave (RLW) equation, Kawahara equation, Benjamin-Bona-Mahoney equation, Bona-Chen equation and several others. These models are derived from first principles under various different hypothesis and approximations. They are all well studied and very well understood.

The dynamics of dispersive shallow water waves, on the other hand, is captured with slightly different models like Rosenau- Kawahara, Rosenau-KdV, and Rosenau-KdV-RLW equations [2, 8, 9, 10, 14].

In particular, the Rosenau-KdV-RLW equation is

$$\partial_t u + a\partial_x u + k\partial_x u^n + b_1\partial^3_{xxx}u + b_2\partial^3_{txx}u + c\partial^5_{txxxx}u = 0, \quad a,\, k,\, b_1,\, b_2,\, c \in \mathbb{R}. \tag{1.1}$$

Here $u(t,x)$ is the nonlinear wave profile. The first term is the linear evolution one, while a is the advection (or drifting) coefficient. The two dispersion coefficients are b_1 and b_2 . The higher order dispersion coefficient is c, while the coefficient of nonlinearity is k where n is the nonlinearity parameter. These are all known and given parameters.

In [14], the authors analyzed (1.1). They got solitary waves, shock waves and singular solitons along with conservation laws.

In the case $n = 2$, $a = 0$, $k = 1$, $b_1 = 1$, $b_2 = -1$, $c = 1$, we have

$$\partial_t u + \partial_x u^2 + \partial^3_{xxx}u - \partial^3_{txx}u + \partial^5_{txxxx}u = 0. \tag{1.2}$$

Choosing $n = 2$, $a = 0$, $k = 1$, $b_2 = b_1 = 0$, $c = 1$, (1.1) reads

$$\partial_t u + \partial_x u^2 + \partial^5_{txxxx}u = 0, \tag{1.3}$$

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which is known as Rosenau equation (see [16, 17]). Existence and uniqueness of solutions for (1.3) has been proved in [13].

Finally, if $n = 2$, $a = 0$, $k = 1$, $b_1 = 1$, $b_2 = 0$, $c = 1$, (1.1) reads

$$\partial_t u + \partial_x u^2 + \partial^3_{xxx} u + \partial^5_{txxxx} u = 0, \tag{1.4}$$

which is known as Rosenau-KdV equation.

In [20], the author discussed the solitary wave solutions and (1.4). In [9], a conservative linear finite difference scheme for the numerical solution for an initial-boundary value problem of Rosenau-KdV equation was considered. In [7, 15], the authors discussed the solitary solutions for (1.4) with solitary ansatz method. The authors also gave two invariants for (1.4). In particular, in [15], the authors studied the two types of soliton solutions, one is a solitary wave and the other is a singular soliton. In [19], the authors proposed an average linear finite difference scheme for the numerical solution of the initial-boundary value problem for (1.4).

If $n = 2$, $a = 0$, $k = 1$, $b_1 = 0$, $b_2 = -1$, $c = 1$, (1.1) reads

$$\partial_t u + \partial_x u^2 - \partial^3_{txx} u + \partial^5_{txxxx} u = 0, \tag{1.5}$$

which is the Rosenau-RLW equation.

In this paper, we analyze (1.2) and (1.5). Arguing as [5], we re-scale the equations as follows

$$\partial_t u + \partial_x u^2 + \beta\partial^3_{xxx} u - \beta\partial^3_{txx} u + \beta^2\partial^5_{txxxx} u = 0, \tag{1.6}$$

$$\partial_t u + \partial_x u^2 - \beta\partial^3_{txx} u + \beta^2\partial^5_{txxxx} u = 0, \tag{1.7}$$

where β is the diffusion parameter.

We are interested in the no high frequency limit, we send $\beta \to 0$ in (1.6) and (1.7). In this way we pass from (1.6) and (1.7) to the equation

$$\partial_t u + \partial_x u^2 = 0 \tag{1.8}$$

which is a scalar conservation law. We prove that, when $\beta \to 0$, the solutions of (1.6) and (1.7) converge to the unique entropy solution (1.8).

The paper is organized in three sections. In Section 2, we prove the convergence of (1.6) to (1.8), while in Section 3, we show how to modify the argument of Section 2 and prove the convergence of (1.7) to (1.8).

2. The Rosenau-KdV-RLW equation.

In this section, we consider (1.6) and augment it with the initial condition

$$u(0, x) = u_0(x), \tag{2.1}$$

on which we assume that

$$u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}). \tag{2.2}$$

We study the dispersion-diffusion limit for (1.6). Therefore, we consider the following fifth order approximation

$$\begin{cases} \partial_t u_{\varepsilon,\beta} + \partial_x u^2_{\varepsilon,\beta} + \beta\partial^3_{xxx} u_{\varepsilon,\beta} - \beta\partial^3_{txx} u_{\varepsilon,\beta} \\ \qquad + \beta^2\partial^5_{txxxx} u_{\varepsilon,\beta} = \varepsilon\partial^2_{xx} u_{\varepsilon,\beta}, & t > 0,\ x \in \mathbb{R}, \\ u_{\varepsilon,\beta}(0, x) = u_{\varepsilon,\beta,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where $u_{\varepsilon,\beta,0}$ is a C^{∞} approximation of u_0 such that

$$
\begin{aligned}
&u_{\varepsilon,\,\beta,\,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbb{R}),\ 1 \le p < 4, \text{ as } \varepsilon,\ \beta \to 0,\\
&\|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \|u_{\varepsilon,\beta,0}\|^4_{L^4(\mathbb{R})} + (\beta+\varepsilon^2)\|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon,\beta>0,\\
&(\beta\varepsilon+\beta\varepsilon^2+\beta^2)\left\|\partial^2_{xx}u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbb{R})} + (\beta^2\varepsilon^2+\beta^3)\left\|\partial^3_{xxx}u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon,\beta>0,\\
&\beta^4\left\|\partial^4_{xxxx}u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon,\beta>0,
\end{aligned}
\tag{2.4}
$$

and C_0 is a constant independent on ε and β.

The main result of this section is the following theorem.

Theorem 2.1. *Assume that* (2.2) *and* (2.4) *hold. If*

$$
\beta = \mathcal{O}(\varepsilon^4), \tag{2.5}
$$

then, there exist two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, *with* $\varepsilon_n, \beta_n \to 0$, *and a limit function*

$$
u \in L^{\infty}(\mathbb{R}^+; L^2(\mathbb{R}) \cap L^4(\mathbb{R})),
$$

such that

$$
u_{\varepsilon_n,\beta_n} \to u \quad \textit{strongly in } L^p_{loc}(\mathbb{R}^+\times\mathbb{R}),\ \textit{for each } 1 \le p < 4; \tag{2.6}
$$

$$
u \quad \textit{is the unique entropy solution of } (1.8). \tag{2.7}
$$

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Lemma 2.1. *For each* $t > 0$,

$$
\begin{aligned}
&\|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad + \beta^2\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int_0^t \|\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \le C_0.
\end{aligned}
\tag{2.8}
$$

In particular, we have

$$
\|u_{\varepsilon,\beta}(t,\cdot)\|_{L^{\infty}(\mathbb{R})} \le C_0\beta^{-\frac{1}{4}}, \tag{2.9}
$$

$$
\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|_{L^{\infty}(\mathbb{R})} \le C_0\beta^{-\frac{3}{4}}. \tag{2.10}
$$

Proof. Multiplying (2.3) by $u_{\varepsilon,\beta}$, we have

$$
\begin{aligned}
&u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta} + 2u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta} + \beta u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}\\
&\qquad - \beta u_{\varepsilon,\beta}\partial^3_{txx}u_{\varepsilon,\beta} + \beta^2 u_{\varepsilon,\beta}\partial^5_{txxxx}u_{\varepsilon,\beta} = \varepsilon u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}.
\end{aligned}
\tag{2.11}
$$

Since

$$
\begin{aligned}
\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta}dx =& \frac{1}{2}\frac{d}{dt}\|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx =& 0,\\
\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx =& \int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}dx = 0,\\
-\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^3_{txx}u_{\varepsilon,\beta}dx =& \frac{\beta}{2}\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^5_{txxxx}u_{\varepsilon,\beta}dx =& -\beta^2\int_{\mathbb{R}}\partial_x u_{\varepsilon,\beta}\partial^4_{txxx}u_{\varepsilon,\beta}dx
\end{aligned}
$$

$$=\frac{\beta^2}{2}\frac{d}{dt}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx = -\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating (2.11) on $\mathbb{R}$, we get

$$\begin{aligned}\frac{d}{dt}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ \beta\frac{d}{dt}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+\beta^2\frac{d}{dt}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 0.\end{aligned}\tag{2.12}$$

(2.8) follows from (2.4), (2.12) and an integration on $(0,t)$.

We prove (2.9). Due to (2.8) and the Hölder inequality,

$$\begin{aligned}u^2_{\varepsilon,\beta}(t,x) =&2\int_{-\infty}^{x} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx \le 2\int_{\mathbb{R}}|u_{\varepsilon,\beta}||\partial_x u_{\varepsilon,\beta}|dx\\ \le&2\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}\le C_0\beta^{-\frac{1}{2}}.\end{aligned}$$

Therefore,

$$|u_{\varepsilon,\beta}(t,x)|\le C_0\beta^{-\frac{1}{4}},$$

which gives (2.9).

Finally, we prove (2.10). Thanks to (2.8) and the Hölder inequality,

$$\begin{aligned}\partial_x u^2_{\varepsilon,\beta}(t,x) =&2\int_{-\infty}^{x}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx \le 2\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx\\ \le&2\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}\le C_0\beta^{-\frac{1}{2}}C_0\beta^{-1}\le C_0\beta^{-\frac{3}{2}}.\end{aligned}$$

Hence,

$$\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\le C_0\beta^{-\frac{3}{4}},$$

that is (2.10). □

Following [3, Lemma 2.2], or [5, Lemma 2.2], or [6, Lemma 4.2], we prove the following result.

Lemma 2.2. *Assume* (2.5). *For each* $t>0$,

i) the family $\{u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ *is bounded in* $L^\infty(\mathbb{R}^+;L^4(\mathbb{R}))$;

ii) the families $\{\varepsilon\partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\sqrt{\beta\varepsilon}\partial_{xx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\varepsilon\sqrt{\beta}\partial_{xx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\varepsilon\beta\partial_{xxx}^3 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta\sqrt{\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta\partial_{xxxx}^4 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ *are bounded in* $L^\infty(\mathbb{R}^+;L^2(\mathbb{R}))$;

iii) the families $\{\sqrt{\beta\varepsilon}\partial_{tx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta\sqrt{\varepsilon}\partial_{txx}^3 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta\sqrt{\beta\varepsilon}\partial_{txxx}^4 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\varepsilon\beta\sqrt{\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\sqrt{\varepsilon}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\varepsilon\sqrt{\varepsilon}\partial_{xx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ *are bounded in* $L^2(\mathbb{R}^+\times\mathbb{R})$.

Moreover,

$$\beta\int_0^t\left\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\partial_{xx}^2 u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^1(\mathbb{R})}ds \le C_0\varepsilon^2,\quad t>0,\tag{2.13}$$

$$\beta^2\int_0^t\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C_0\varepsilon^5,\quad t>0.\tag{2.14}$$

Proof. Let A, B, C be some positive constants which will be specified later. Multiplying (2.3) by

$$u^3_{\varepsilon,\beta} - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta},$$

we have

$$\begin{aligned}
&\left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}\\
&\qquad + 2\left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right) u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\\
&\qquad + \beta\left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_{xxx}^3 u_{\varepsilon,\beta}\\
&\qquad - \beta\left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_{txx}^3 u_{\varepsilon,\beta}\\
&\qquad + \beta^2\left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_{txxxx}^5 u_{\varepsilon,\beta}\\
&\quad = \varepsilon\left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_{xx}^2 u_{\varepsilon,\beta}.
\end{aligned}\tag{2.15}$$

We observe that

$$\begin{aligned}
&\int_{\mathbb{R}}\left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}dx\\
&\qquad = \frac{1}{4}\frac{d}{dt}\|u_{\varepsilon,\beta}(t,\cdot)\|_{L^4(\mathbb{R})}^4 + A\beta\varepsilon\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\qquad\quad + \frac{B\varepsilon^2}{2}\frac{d}{dt}\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \frac{C\beta^2}{2}\frac{d}{dt}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}\tag{2.16}$$

We have that

$$\begin{aligned}
&2\int_{\mathbb{R}}\left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right) u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx\\
&\qquad = -2A\beta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx - 2B\varepsilon^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx\\
&\qquad\quad - 2C\beta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_{xxx}^3 u_{\varepsilon,\beta}dx - 2C\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}dx.
\end{aligned}\tag{2.17}$$

Since

$$\begin{aligned}
-2C\beta^2\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_{xxx}^3 u_{\varepsilon,\beta} - 2C\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}dx&\\
=5C\beta^2\int_{\mathbb{R}}(\partial_{xx}^2 u_{\varepsilon,\beta})^2\partial_x u_{\varepsilon,\beta}dx&\\
=-\frac{5\beta^2}{2}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_{xxx}^3 u_{\varepsilon,\beta}dx,&
\end{aligned}\tag{2.18}$$

it follows from (2.17) and (2.18) that

$$\begin{aligned}
&2\int_{\mathbb{R}}\left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right) u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx\\
&\qquad = -2A\beta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx - 2B\varepsilon^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx\\
&\qquad\quad - \frac{5C\beta^2}{2}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2\partial_{xxx}^3 u_{\varepsilon,\beta}dx
\end{aligned}\tag{2.19}$$

.

We observe

$$\begin{aligned}
&\beta\int_{\mathbb{R}}\left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta} - B\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} + C\beta^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_{xxx}^3 u_{\varepsilon,\beta}dx\\
&\qquad = -3\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx - A\beta^2\varepsilon\int_{\mathbb{R}}\partial_{xxx}^3 u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx.
\end{aligned}\tag{2.20}$$

We get

$$
\begin{aligned}
&-\beta\int_{\mathbb{R}}\left(u_{\varepsilon,\beta}^3-A\beta\varepsilon\partial_{txx}^3u_{\varepsilon,\beta}-B\varepsilon^2\partial_{xx}^2u_{\varepsilon,\beta}+C\beta^2\partial_{xxxx}^4u_{\varepsilon,\beta}\right)\partial_{txx}^3u_{\varepsilon,\beta}dx\\
&\qquad=3\beta\int_{\mathbb{R}}u_{\varepsilon,\beta}^2\partial_xu_{\varepsilon,\beta}\partial_{tx}^2u_{\varepsilon,\beta}dx+A\beta^2\varepsilon\left\|\partial_{txx}^3u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\qquad\quad+\frac{B\beta\varepsilon^2}{2}\frac{d}{dt}\left\|\partial_{xx}^2u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{C\beta^3}{2}\frac{d}{dt}\left\|\partial_{xxx}^3u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}\tag{2.21}
$$

We have that

$$
\begin{aligned}
&\beta^2\int_{\mathbb{R}}\left(u_{\varepsilon,\beta}^3-A\beta\varepsilon\partial_{txx}^3u_{\varepsilon,\beta}-B\varepsilon^2\partial_{xx}^2u_{\varepsilon,\beta}+C\beta^2\partial_{xxxx}^4u_{\varepsilon,\beta}\right)\partial_{txxxx}^5u_{\varepsilon,\beta}dx\\
&\qquad=-3\beta^2\int_{\mathbb{R}}u_{\varepsilon,\beta}^2\partial_xu_{\varepsilon,\beta}\partial_{txxx}^4u_{\varepsilon,\beta}dx+A\beta^3\varepsilon\left\|\partial_{txxx}^4u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\qquad\quad+\frac{B\beta^2\varepsilon^2}{2}\frac{d}{dt}\left\|\partial_{xxx}^3u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{C\beta^4}{2}\frac{d}{dt}\left\|\partial_{xxxx}^4u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}\tag{2.22}
$$

Moreover,

$$
\begin{aligned}
&\varepsilon\int_{\mathbb{R}}\left(u_{\varepsilon,\beta}^3-A\beta\varepsilon\partial_{txx}^3u_{\varepsilon,\beta}-B\varepsilon^2\partial_{xx}^2u_{\varepsilon,\beta}+C\beta^2\partial_{xxxx}^4u_{\varepsilon,\beta}\right)\partial_{xx}^2u_{\varepsilon,\beta}dx\\
&\qquad=-3\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_xu_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2-\frac{A\beta\varepsilon}{2}\frac{d}{dt}\left\|\partial_{xx}^2u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\qquad\quad-\varepsilon^3B\left\|\partial_{xx}^2u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2-\beta^2\varepsilon C\left\|\partial_{xxx}^3u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}\tag{2.23}
$$

It follows from (2.16), (2.19), (2.20), (2.21), (2.22), (2.23), and an integration of (2.15) on $\mathbb{R}$ that

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^4(\mathbb{R})}^4+\frac{\left(A\beta\varepsilon+B\beta\varepsilon^2+C\beta^2\right)}{2}\left\|\partial_{xx}^2u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\right)\\
&\quad+\frac{d}{dt}\left(\frac{B\varepsilon^2}{2}\left\|\partial_xu_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{\left(B\beta^2\varepsilon^2+C\beta^3\right)}{2}\left\|\partial_{xxx}^3u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\right)\\
&\quad+\frac{C\beta^4}{2}\frac{d}{dt}\left\|\partial_{xxxx}^4u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+A\beta\varepsilon\left\|\partial_{tx}^2u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\quad+A\beta^2\varepsilon\left\|\partial_{txx}^3u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+A\beta^3\varepsilon\left\|\partial_{txxx}^4u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad+3\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_xu_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\varepsilon^3B\left\|\partial_{xx}^2u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad+\beta^2\varepsilon C\left\|\partial_{xxx}^3u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&=2A\beta\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_xu_{\varepsilon,\beta}\partial_{txx}^3u_{\varepsilon,\beta}dx+2B\varepsilon^2\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_xu_{\varepsilon,\beta}\partial_{xx}^2u_{\varepsilon,\beta}dx\\
&\quad\frac{5C\beta^2}{2}\int_{\mathbb{R}}(\partial_xu_{\varepsilon,\beta})^2\partial_{xxx}^3u_{\varepsilon,\beta}dx-3\beta\int_{\mathbb{R}}u_{\varepsilon,\beta}^2\partial_xu_{\varepsilon,\beta}\partial_{xx}^2u_{\varepsilon,\beta}dx\\
&\quad+A\beta^2\varepsilon\int_{\mathbb{R}}\partial_{xxx}^3u_{\varepsilon,\beta}\partial_{txx}^3u_{\varepsilon,\beta}-3\beta\int_{\mathbb{R}}u_{\varepsilon,\beta}^2\partial_xu_{\varepsilon,\beta}\partial_{tx}^2u_{\varepsilon,\beta}dx\\
&\quad+3\beta^2\int_{\mathbb{R}}u_{\varepsilon,\beta}^2\partial_xu_{\varepsilon,\beta}\partial_{txxx}^4u_{\varepsilon,\beta}dx.
\end{aligned}\tag{2.24}
$$

Due to the Young inequality,

$$
\begin{aligned}
2A\beta\varepsilon\left|\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx\right| &\leq A\varepsilon\int_{\mathbb{R}}|2u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}|\,|\beta\partial_{txx}^3 u_{\varepsilon,\beta}|dx\\
&\leq 2A\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{A\beta^2\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2B\varepsilon^2\left|\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx\right| &\leq \int_{\mathbb{R}}\left|\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\right|\left|\varepsilon^{\frac{3}{2}}2B\partial_{xx}^2 u_{\varepsilon,\beta}\right|dx\\
&\leq\frac{\varepsilon}{2}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2B^2\varepsilon^3\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.25}
$$

Hence, from (2.24),

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\frac{\left(A\beta\varepsilon+B\beta\varepsilon^2+C\beta^2\right)}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\frac{d}{dt}\left(\frac{B\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\left(B\beta^2\varepsilon^2+C\beta^3\right)}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\frac{C\beta^4}{2}\frac{d}{dt}\left\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+A\beta\varepsilon\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{A\beta^2\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+A\beta^3\varepsilon\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\left(\frac{5}{2}-2A\right)\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\varepsilon C\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\left(B-2B^2\right)\varepsilon^3\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq\frac{5C\beta^2}{2}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2|\partial_{xxx}^3 u_{\varepsilon,\beta}|dx+3\beta\int_{\mathbb{R}}u_{\varepsilon,\beta}^2|\partial_x u_{\varepsilon,\beta}|\partial_{xx}^2 u_{\varepsilon,\beta}|dx\\
&\quad+A\beta^2\varepsilon\int_{\mathbb{R}}|\partial_{xxx}^3 u_{\varepsilon,\beta}||\partial_{txx}^3|dx+3\beta\int_{\mathbb{R}}u_{\varepsilon,\beta}^2|\partial_x u_{\varepsilon,\beta}||\partial_{tx}^2 u_{\varepsilon,\beta}|dx\\
&\quad+3\beta^2\int_{\mathbb{R}}u_{\varepsilon,\beta}^2|\partial_x u_{\varepsilon,\beta}||\partial_{txxx}^4 u_{\varepsilon,\beta}|dx.
\end{aligned}
\tag{2.26}
$$

From (2.5), we have

$$
\beta\leq D^2\varepsilon^4,
\tag{2.27}
$$

where D is a positive constant which will be specified later. It follows from (2.10), (2.27) and the Young inequality that

$$
\begin{aligned}
\frac{5C\beta^2}{2}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^2|\partial_{xxx}^3 u_{\varepsilon,\beta}|dx &= \beta^2 C\int_{\mathbb{R}}\frac{5}{2\varepsilon^{\frac{1}{2}}}(\partial_x u_{\varepsilon,\beta})^2\left|\varepsilon^{\frac{1}{2}}\partial_{xxx}^3 u_{\varepsilon,\beta}\right|\\
&\leq\frac{25C\beta^2}{8\varepsilon}\int_{\mathbb{R}}(\partial_x u_{\varepsilon,\beta})^4dx+\frac{C\beta^2\varepsilon}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq\frac{25C}{8\varepsilon}\beta^2\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{C\beta^2\varepsilon}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq\frac{C_0\beta^{\frac{1}{2}}}{\varepsilon}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{C\beta^2\varepsilon}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C_0 D\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{C\beta^2\varepsilon}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.28}
$$

Due to (2.9), (2.27) and the Young inequality,

$$
\begin{aligned}
3\beta\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx}u_{\varepsilon,\beta}|dx &\le 3\beta\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx}u_{\varepsilon,\beta}|dx\\
&\le C_0\beta^{\frac{1}{2}}\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx}u_{\varepsilon,\beta}|dx \le \int_{\mathbb{R}}\left|\varepsilon^{\frac{1}{2}}\partial_x u_{\varepsilon,\beta}\right|\left|C_0 D\varepsilon^{\frac{3}{2}}\partial^2_{xx}u_{\varepsilon,\beta}\right|dx\\
&\le \varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D^2C_0^2\varepsilon^3\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.29}
$$

Thanks to the Young inequality,

$$
\begin{aligned}
A\beta^2\varepsilon\int_{\mathbb{R}}|\partial^3_{xxx}u_{\varepsilon,\beta}||\partial^3_{txx}u_{\varepsilon,\beta}|dx &= A\beta^2\varepsilon\int_{\mathbb{R}}\left|2\partial^3_{xxx}u_{\varepsilon,\beta}\right|\left|\frac{1}{2}\partial^3_{txx}u_{\varepsilon,\beta}\right|dx\\
&\le 2A\beta^2\varepsilon\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{A\beta^2\varepsilon}{8}\left\|\partial^3_{txx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.30}
$$

It follows from (2.9), (2.27) and the Young inequality that

$$
\begin{aligned}
3\beta\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial^2_{tx}u_{\varepsilon,\beta}|dx &= \beta\int_{\mathbb{R}}\left|\frac{3u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\varepsilon^{\frac{1}{2}}A^{\frac{1}{2}}}\right|\left|\varepsilon^{\frac{1}{2}}A^{\frac{1}{2}}\partial^2_{tx}u_{\varepsilon,\beta}\right|dx\\
&\le \frac{9\beta}{2\varepsilon A}\int_{\mathbb{R}} u^4_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2dx + \frac{\beta\varepsilon A}{2}\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{9}{2\varepsilon A}\beta\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \frac{\beta\varepsilon A}{2}\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C_0\beta^{\frac{1}{2}}}{\varepsilon A}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta\varepsilon A}{2}\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C_0D}{A}\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta\varepsilon A}{2}\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.31}
$$

Again by (2.9), (2.27) and the Young inequality,

$$
\begin{aligned}
3\beta^2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial^4_{txxx}u_{\varepsilon,\beta}|dx &= \int_{\mathbb{R}}\left|\frac{3\beta^{\frac{1}{2}}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\varepsilon^{\frac{1}{2}}A^{\frac{1}{2}}}\right|\left|\beta^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}A^{\frac{1}{2}}\partial^4_{txxx}u_{\varepsilon,\beta}\right|dx\\
&\le \frac{9\beta}{2\varepsilon A}\int_{\mathbb{R}} u^4_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2dx + \frac{\beta^3\varepsilon A}{2}\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{9}{2\varepsilon A}\beta\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \frac{\beta^3\varepsilon A}{2}\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C_0\beta^{\frac{1}{2}}}{\varepsilon A}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^3\varepsilon A}{2}\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C_0D}{A}\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^3\varepsilon A}{2}\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.32}
$$

From (2.26), (2.28), (2.29), (2.30), (2.31) and (2.32), we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\frac{\left(A\beta\varepsilon+B\beta\varepsilon^2+C\beta^2\right)}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\frac{d}{dt}\left(\frac{B\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\left(B\beta^2\varepsilon^2+C\beta^3\right)}{2}\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\frac{C\beta^4}{2}\frac{d}{dt}\left\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta\varepsilon A}{2}\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{3A\beta^2\varepsilon}{8}\left\|\partial^3_{txx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{A\beta^3\varepsilon}{2}\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\beta^2\varepsilon\left(\frac{C}{2}-2A\right)\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\left(\frac{5}{2}-2A-\frac{C_0D}{A}\right)\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\left(B-2B^2-D^2C_0^2\right)\varepsilon^3\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C_0\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.33}
$$

We search A, B, C such that

$$
\begin{cases}
\frac{5}{2}-2A-\frac{C_0D}{A}>0,\\
B-2B^2-D^2C_0^2>0,\\
\frac{C}{2}-2A>0,
\end{cases}
$$

that is

$$
\begin{cases}
4A^2-5A+2C_0D<0,\\
2B^2-B-D^2C_0^2<0,\\
C>4A.
\end{cases}
\tag{2.34}
$$

We choose

$$
C=6A. \tag{2.35}
$$

The first inequality of (2.34) admits a solution, if

$$
25-32C_0D>0,
$$

that is

$$
D<\frac{25}{32C_0}. \tag{2.36}
$$

The second inequality of (2.34) admits a solution, if

$$
1-8D^2C_0^2>0,
$$

that is

$$
D<\frac{\sqrt{2}}{4C_0}. \tag{2.37}
$$

It follows from (2.36) and (2.37) that

$$
D<\min\left\{\frac{25}{32C_0},\frac{\sqrt{2}}{4C_0}\right\}=\frac{\sqrt{2}}{4C_0}. \tag{2.38}
$$

Therefore, from (2.34), (2.35) and (2.38), we have that there exist $0 < A_1 < A_2$ and $0 < B_1 < B_2$, such that choosing

$$A_1 < A < A_2, \quad B_1 < B < B_2, \quad C = 6A, \tag{2.39}$$

(2.34) holds.

(2.33) and (2.34) give

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{\left(A\beta\varepsilon + B\beta\varepsilon^2 + 6A\beta^2\right)}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + \frac{d}{dt}\left(\frac{B\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\left(B\beta^2\varepsilon^2 + 6A\beta^3\right)}{2}\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + 3A\beta^4\frac{d}{dt}\left\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta\varepsilon A}{2}\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \frac{3A\beta^2\varepsilon}{8}\left\|\partial^3_{txx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{A\beta^3\varepsilon}{2}\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \beta^2\varepsilon A\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon K_1\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \varepsilon^3 K\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C_0\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

for some $K_1,\, K_2 > 0$.

(2.4), (2.8) and an integration on $(0,t)$ give

$$\begin{aligned}
&\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{\left(A\beta\varepsilon + B\beta\varepsilon^2 + 6A\beta^2\right)}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \frac{B\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\left(B\beta^2\varepsilon^2 + 6A\beta^3\right)}{2}\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + 3A\beta^4\left\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta\varepsilon A}{2}\int_0^t\left\|\partial^2_{tx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad + \frac{3A\beta^2\varepsilon}{8}\int_0^t\left\|\partial^3_{txx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds + \frac{A\beta^3\varepsilon}{2}\int_0^t\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad + \beta^2\varepsilon A\int_0^t\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds + \varepsilon K_1\int_0^t\left\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad + \varepsilon^3 K_2\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\le C_0 + C_0\varepsilon\int_0^t\left\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le C_0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^4(\mathbb{R})} \le& C_0,\\
\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \le& C_0,\\
\sqrt{\beta\varepsilon}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \le& C_0,\\
\varepsilon\sqrt{\beta}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \le& C_0,\\
\beta\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \le& C_0,\\
\beta\varepsilon\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \le& C_0,
\end{aligned}$$

$$\begin{aligned}
\beta\sqrt{\beta}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \le& C_0,\\
\beta\left\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \le& C_0,\\
\beta\varepsilon\int_0^t\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le& C_0,\\
\beta^2\varepsilon\int_0^t\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le& C_0,\\
\beta^3\varepsilon\int_0^t\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le& C_0,\\
\beta^2\varepsilon\int_0^t\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le& C_0,\\
\varepsilon\int_0^t\left\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le& C_0,\\
\varepsilon^3\int_0^t\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le& C_0,
\end{aligned}$$

for every $t > 0$.

Arguing as [3, Lemma 2.2], we have (2.13) and (2.14). □

To prove Theorem 2.1. The following technical lemma is needed [12].

Lemma 2.3. *Let Ω be a bounded open subset of $\mathbb{R}^2$. Suppose that the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

Moreover, we consider the following definition.

Definition 2.1. *A pair of functions (η, q) is called an entropy–entropy flux pair if $\eta : \mathbb{R} \to \mathbb{R}$ is a C^2 function and $q : \mathbb{R} \to \mathbb{R}$ is defined by*

$$q(u) = \int_0^u A\xi\eta'(\xi)d\xi.$$

An entropy-entropy flux pair $(\eta,\ q)$ is called convex/compactly supported if, in addition, η is convex/compactly supported.

Following [11], we prove Theorem 2.1.

Proof of Theorem 2.1. Let us consider a compactly supported entropy–entropy flux pair (η, q). Multiplying (2.3) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}
\partial_t\eta(u_{\varepsilon,\beta}) + \partial_x q(u_{\varepsilon,\beta}) =&\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xx}^2 u_{\varepsilon,\beta} - \beta\eta'(u_{\varepsilon,\beta})\partial_{xxx}^3 u_{\varepsilon,\beta}\\
&-\beta\eta'(u_{\varepsilon,\beta})\partial_{txx}^3 u_{\varepsilon,\beta} + \beta^2\eta'(u_{\varepsilon,\beta})\partial_{txxxx}^5 u_{\varepsilon,\beta}\\
=&I_{1,\,\varepsilon,\,\beta} + I_{2,\,\varepsilon,\,\beta} + I_{3,\,\varepsilon,\,\beta} + I_{4,\,\varepsilon,\,\beta} + I_{5,\,\varepsilon,\,\beta}\\
&+ I_{6,\,\varepsilon,\,\beta} + I_{7,\,\varepsilon,\,\beta} + I_{8,\,\varepsilon,\,\beta},
\end{aligned}$$

where

$$
\begin{aligned}
I_{1,\varepsilon,\beta} &= \partial_x(\varepsilon\eta'(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}),\\
I_{2,\varepsilon,\beta} &= -\varepsilon\eta''(u_{\varepsilon,\beta})(\partial_x u_{\varepsilon,\beta})^2,\\
I_{3,\varepsilon,\beta} &= \partial_x(-\beta\eta'(u_{\varepsilon,\beta})\partial_{xx}^2 u_{\varepsilon,\beta}),\\
I_{4,\varepsilon,\beta} &= \beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta},\\
I_{5,\varepsilon,\beta} &= \partial_x(-\beta\eta'(u_{\varepsilon,\beta})\partial_{tx}^2 u_{\varepsilon,\beta}),\\
I_{6,\varepsilon,\beta} &= \beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{tx}^2 u_{\varepsilon,\beta},\\
I_{7,\varepsilon,\beta} &= \partial_x(\beta^2\eta'(u_{\varepsilon,\beta})\partial_{txxx}^4 u_{\varepsilon,\beta}),\\
I_{8,\varepsilon,\beta} &= -\beta^2\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{txxx}^4 u_{\varepsilon,\beta}.
\end{aligned}
\tag{2.40}
$$

Fix $T>0$. Arguing as [4, Lemma 3.2], we have that $I_{1,\varepsilon,\beta}\to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, and $\{I_{2,\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$.
Arguing as [3, Theorem 1.1], we get $I_{3,\varepsilon,\beta}\to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, and $I_{4,\varepsilon,\beta}\to 0$ in $L^1((0,T)\times\mathbb{R})$.
We claim that

$$I_{5,\varepsilon,\beta}\to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}),\ T>0, \text{ as } \varepsilon\to 0.$$

By (2.5) and Lemma 2.2,

$$
\begin{aligned}
&\left\|\beta\eta'(u_{\varepsilon,\beta})\partial_{tx}^2 u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le \left\|\eta'\right\|^2_{L^\infty(\mathbb{R})}\beta^2\int_0^T\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}dt\\
&\quad= \left\|\eta'\right\|^2_{L^\infty(\mathbb{R})}\frac{\beta^2\varepsilon}{\varepsilon}\int_0^T\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}dt\\
&\quad\le C_0\left\|\eta'\right\|^2_{L^\infty(\mathbb{R})}\frac{\beta}{\varepsilon}\le C_0\left\|\eta'\right\|^2_{L^\infty(\mathbb{R})}\varepsilon^3\to 0.
\end{aligned}
$$

We have that

$$I_{6,\varepsilon,\beta}\to 0 \quad \text{in } L^1((0,T)\times\mathbb{R}),\ T>0, \text{ as } \varepsilon\to 0.$$

Due to (2.5), Lemmas 2.1, 2.2 and the Hölder inequality,

$$
\begin{aligned}
&\left\|\beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{tx}^2 u_{\varepsilon,\beta}\right\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad\le \left\|\eta''\right\|_{L^\infty(\mathbb{R})}\beta\int_0^T\!\!\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}\partial_{tx}^2 u_{\varepsilon,\beta}|dtdx\\
&\quad\le \left\|\eta''\right\|_{L^\infty(\mathbb{R})}\frac{\beta\varepsilon}{\varepsilon}\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial_t u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\frac{\beta^{\frac{1}{2}}}{\varepsilon}\le C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\varepsilon\to 0.
\end{aligned}
$$

We claim that

$$I_{7,\varepsilon,\beta}\to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}),\ T>0, \text{ as } \varepsilon\to 0.$$

By (2.5) and Lemma 2.2,

$$
\begin{aligned}
&\left\|\beta^2\eta'(u_{\varepsilon,\beta})\partial_{txxx}^4 u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad\le \beta^4\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}\\
&\quad= \left\|\eta'\right\|_{L^\infty(\mathbb{R})}\frac{\beta^4\varepsilon}{\varepsilon}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}
\end{aligned}
$$

$$\leq C_0 \left\|\eta'\right\|_{L^\infty(\mathbb{R})} \frac{\beta}{\varepsilon} \leq C_0 \left\|\eta'\right\|_{L^\infty(\mathbb{R})} \varepsilon^3 \to 0.$$

We have that

$$I_{8,\varepsilon,\beta} \to 0 \quad \text{in } L^1((0,T)\times\mathbb{R}),\, T>0, \text{ as } \varepsilon \to 0.$$

Thanks to (2.5), Lemmas 2.1, 2.2 and the Hölder inequality,

$$\begin{aligned}
&\left\|\beta^2\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{txxx}^4 u_{\varepsilon,\beta}\right\|_{L^1((0,T)\times\mathbb{R})}\\
&\quad\leq \beta^2 \left\|\eta''\right\|_{L^\infty(\mathbb{R})}\int_0^T\!\!\int_{\mathbb{R}} |\partial_x u_{\varepsilon,\beta}\partial_{txxx}^4 u_{\varepsilon,\beta}| dsdx\\
&\quad\leq \left\|\eta''\right\|_{L^\infty(\mathbb{R})}\frac{\beta^2\varepsilon}{\varepsilon}\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\\
&\quad\leq C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\frac{\beta^{\frac{1}{2}}}{\varepsilon} \leq C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\varepsilon \to 0.
\end{aligned}$$

Therefore, (2.6) follows from Lemma 2.3 and the L^p compensated compactness of [18].

To have (2.7), we begin by proving that u is a distributional solution of (1.8). Let $\phi \in C^\infty(\mathbb{R}^2)$ be a test function with compact support. We have to prove that

$$\int_0^\infty\!\!\int_{\mathbb{R}} \left(u\partial_t\phi + u^2\partial_x\phi\right) dtdx + \int_{\mathbb{R}} u_0(x)\phi(0,x)dx = 0. \tag{2.41}$$

We define

$$u_{\varepsilon_n,\beta_n} := u_n. \tag{2.42}$$

We have that

$$\begin{aligned}
&\int_0^\infty\!\!\int_{\mathbb{R}} \left(u_n\partial_t\phi + u_n^2\partial_x\phi\right) dtdx + \int_{\mathbb{R}} u_{0,n}(x)\phi(0,x)dx\\
&\quad+\varepsilon_n\int_0^\infty\!\!\int_{\mathbb{R}} u_n\partial_{xx}^2\phi dtdx + \varepsilon_n\int_0^\infty u_{0,n}(x)\partial_{xx}^2\phi(0,x)dx\\
&\quad+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}} u_n\partial_{xxx}^3\phi dtdx + \beta_n\int_0^\infty u_{0,n}(x)\partial_{xxx}^3\phi(0,x)dx\\
&\quad-\beta_n\int_0^\infty\!\!\int_{\mathbb{R}} u_n\partial_{txx}^3\phi dtds - \beta_n\int_0^\infty u_{0,n}(x)\partial_{txx}^3\phi(0,x)dx\\
&\quad+\beta_n^2\int_0^\infty\!\!\int_{\mathbb{R}} u_n\partial_{txxxx}^5\phi dtds - \beta_n\int_0^\infty u_{0,n}(x)\partial_{txxxx}^5\phi(0,x)dx = 0.
\end{aligned}$$

Therefore, (2.41) follows from (2.4) and (2.6).

We conclude by proving that u is the unique entropy solution of (1.8). Fix $T>0$. Let us consider a compactly supported entropy–entropy flux pair (η,q), and $\phi\in C_c^\infty((0,\infty)\times\mathbb{R})$ a non–negative function. We have to prove that

$$\int_0^\infty\!\!\int_{\mathbb{R}} (\partial_t\eta(u) + \partial_x q(u))\phi dtdx \leq 0. \tag{2.43}$$

We have

$$\begin{aligned}
&\int_0^\infty\!\!\int_{\mathbb{R}} (\partial_t\eta(u_n) + \partial_x q(u_n))\phi dtdx\\
&\quad= \varepsilon_n\int_0^\infty\!\!\int_{\mathbb{R}} \partial_x(\eta'(u_n)\partial_x u_n)\phi dtdx - \varepsilon_n\int_0^\infty\!\!\int_{\mathbb{R}} \eta''(u_n)(\partial_x u_n)^2\phi dtdx\\
&\quad\quad-\beta_n\int_0^\infty\!\!\int_{\mathbb{R}} \partial_x(\eta'(u_n)\partial_{xx}^2 u_n)\phi dtdx
\end{aligned}$$

$$
\begin{aligned}
&+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}\eta''(u_n)\partial_x u_n\partial_{xx}^2u_n\phi dtdx-\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}\partial_x(\eta'(u_n)\partial_{tx}^2u_n)\phi dtdx\\
&+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}\eta''(u_n)\partial_x u_n\partial_{tx}^2u_n\phi dtdx+\beta_n^2\int_0^\infty\!\!\int_{\mathbb{R}}\partial_x(\eta'(u_n)\partial_{txxx}^4u_n)\phi dtdx\\
&-\beta_n^2\int_0^\infty\!\!\int_{\mathbb{R}}\eta''(u_n)\partial_x u_n\partial_{txxx}^4u_n\phi dtdx\\
\le&-\varepsilon_n\int_0^\infty\!\!\int_{\mathbb{R}}\eta'(u_n)\partial_x u_n\partial_x\phi dtdx+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}\eta'(u_n)\partial_{xx}^2u_n\partial_x\phi dtdx\\
&+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}\eta''(u_n)\partial_x u_n\partial_{xx}^2u_n\phi dtdx+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}\eta'(u_n)\partial_{tx}^2u_n\partial_x\phi dtdx\\
&+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}\eta''(u_n)\partial_x u_n\partial_{tx}^2u_n\phi dtdx-\beta_n^2\int_0^\infty\!\!\int_{\mathbb{R}}\eta'(u_n)\partial_{txxx}^4u_n\partial_x\phi dtdx\\
&-\beta_n^2\int_0^\infty\!\!\int_{\mathbb{R}}\eta''(u_n)\partial_x u_n\partial_{txxx}^4u_n\phi dtdx\\
\le&\ \varepsilon_n\int_0^\infty\!\!\int_{\mathbb{R}}|\eta'(u_n)||\partial_x u_n||\partial_x\phi|dtdx+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}|\eta'(u_n)||\partial_{xx}^2u_n||\partial_x\phi|dtdx\\
&+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}|\eta''(u_n)||\partial_x u_n||\partial_{xx}^2u_n||\phi|dtdx+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}|\eta'(u_n)|||\partial_{tx}^2u_n||\partial_x\phi|dtdx\\
&+\beta_n\int_0^\infty\!\!\int_{\mathbb{R}}|\eta''(u_n)||\partial_x u_n||\partial_{tx}^2u_n||\phi|dtdx+\beta_n^2\int_0^\infty\!\!\int_{\mathbb{R}}|\eta'(u_n)||\partial_{txxx}^4u_n||\partial_x\phi|dtdx\\
&+\beta_n^2\int_0^\infty\!\!\int_{\mathbb{R}}|\eta''(u_n)||\partial_x u_n||\partial_{txxx}^4u_n|\phi dtdx\\
\le&\ \varepsilon_n\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\|\partial_x u_n\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\|\partial_x\phi\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\\
&+\beta_n\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{xx}^2u_n\right\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\|\partial_x\phi\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\\
&+\beta_n\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\left\|\partial_x u_n\partial_{xx}^2u_n\right\|_{L^1(\mathrm{supp}\,(\phi))}\\
&+\beta_n\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{tx}^2u_n\right\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\|\partial_x\phi\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\\
&+\beta_n\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}}\left\|\partial_x u_n\partial_{tx}^2u_n\right\|_{L^1(\mathrm{supp}\,(\phi))}\\
&+\beta_n^2\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{txxx}^4u_n\right\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\|\partial_x\phi\|_{L^2(\mathrm{supp}\,(\partial_x\phi))}\\
&+\beta_n^2\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\left\|\partial_x u_n\partial_{txxx}^4u_n\right\|_{L^1(\mathrm{supp}\,(\phi))}\\
\le&\ \varepsilon_n\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\|\partial_x u_n\|_{L^2((0,T)\times\mathbb{R})}\|\partial_x\phi\|_{L^2((0,T)\times\mathbb{R})}\\
&+\beta_n\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{xx}^2u_n\right\|_{L^2((0,T)\times\mathbb{R})}\|\partial_x\phi\|_{L^2((0,T)\times\mathbb{R})}\\
&+\beta_n\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\left\|\partial_x u_n\partial_{xx}^2u_n\right\|_{L^1((0,T)\times\mathbb{R})}\\
&+\beta_n\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{tx}^2u_n\right\|_{L^2((0,T)\times\mathbb{R})}\|\partial_x\phi\|_{L^2((0,T)\times\mathbb{R})}\\
&+\beta_n\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\left\|\partial_x u_n\partial_{tx}^2u_n\right\|_{L^1((0,T)\times\mathbb{R})}\\
&+\beta_n^2\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial_{txxx}^4u_n\right\|_{L^2((0,T)\times\mathbb{R})}\|\partial_x\phi\|_{L^2((0,T)\times\mathbb{R})}\\
&+\beta_n^2\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\left\|\partial_x u_n\partial_{txxx}^4u_n\right\|_{L^1((0,T)\times\mathbb{R})}.
\end{aligned}
$$

(2.43) follows from (2.5), (2.6), Lemmas 2.1 and 2.2. ∎

3. The Rosenau-RLW equation

In this section, we consider (1.7) and augment (1.7) with the initial condition

$$u(0,x)=u_0(x), \tag{3.1}$$

on which we assume that

$$u_0\in L^2(\mathbb{R})\cap L^4(\mathbb{R}). \tag{3.2}$$

We study the dispersion-diffusion limit for (1.7). Therefore, we consider the following fifth order problem

$$\begin{cases}\partial_t u_{\varepsilon,\beta}+\partial_x u_{\varepsilon,\beta}^2-\beta\partial_{txx}^3 u_{\varepsilon,\beta}+\beta^2\partial_{txxxx}^5 u_{\varepsilon,\beta}=\varepsilon\partial_{xx}^2 u_{\varepsilon,\beta}, & t>0,\ x\in\mathbb{R},\\ u_{\varepsilon,\beta}(0,x)=u_{\varepsilon,\beta,0}(x), & x\in\mathbb{R},\end{cases} \tag{3.3}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned}&u_{\varepsilon,\beta,0}\to u_0\quad \text{in } L^p_{loc}(\mathbb{R}),\ 1\le p<4, \text{ as } \varepsilon,\ \beta\to 0,\\ &\|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})}+\|u_{\varepsilon,\beta,0}\|^4_{L^4(\mathbb{R})}+\beta\varepsilon^2\left\|\partial_{xx}^2 u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbb{R})}\le C_0,\quad \varepsilon,\beta>0,\end{aligned} \tag{3.4}$$

and C_0 is a constant independent on ε and β.

The main result of this section is the following theorem.

Theorem 3.1. *Assume that* (3.2) *and* (2.4) *hold. If*

$$\beta=\mathcal{O}(\varepsilon^4), \tag{3.5}$$

then, there exist two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, *with* $\varepsilon_n,\beta_n\to 0$, *and a limit function*

$$u\in L^\infty(\mathbb{R}^+;L^2(\mathbb{R})\cap L^4(\mathbb{R})),$$

such that

$$u_{\varepsilon_n,\beta_n}\to u\quad \text{strongly in } L^p_{loc}(\mathbb{R}^+\times\mathbb{R}),\ \text{for each } 1\le p<4; \tag{3.6}$$

$$u\quad \text{is the unique entropy solution of (1.8).} \tag{3.7}$$

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend on the initial data.

We begin by observing that Lemma 2.1 holds also for (3.3).

Following [3, Lemma 2.2], or [5, Lemma 2.2], or [6, Lemma 4.2], we prove the following result.

Lemma 3.1. *Assume* (3.5). *For each* $t>0$,

i) the family $\{u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ *is bounded in* $L^\infty(\mathbb{R}^+;L^4(\mathbb{R}))$;
ii) the family $\{\varepsilon\sqrt{\beta}\partial_{xx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ *is bounded in* $L^\infty(\mathbb{R}^+;L^2(\mathbb{R}))$;
iii) the families $\{\sqrt{\beta\varepsilon}\partial_{tx}^2 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta\sqrt{\varepsilon}\partial_{txx}^3 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta\sqrt{\beta\varepsilon}\partial_{txxx}^4 u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\sqrt{\varepsilon}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ *are bounded in* $L^2(\mathbb{R}^+\times\mathbb{R})$.

Proof. Let A be a positive constant which will be specified later. Multiplying (3.3) by $u_{\varepsilon,\beta}^3-A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}$, we have

$$\begin{aligned}&\left(u^3_{\varepsilon,\beta}-A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}+2\left(u^3_{\varepsilon,\beta}-A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\\ &\qquad-\beta\left(u^3_{\varepsilon,\beta}-A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}\right)\partial_{txx}^3 u_{\varepsilon,\beta}+\beta^2\left(u^3_{\varepsilon,\beta}-A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}\right)\partial_{txxxx}^5 u_{\varepsilon,\beta}\\ &\quad=\varepsilon\left(u^3_{\varepsilon,\beta}-A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}\right)\partial_{xx}^2 u_{\varepsilon,\beta}.\end{aligned} \tag{3.8}$$

Since

$$\begin{aligned}
\int_{\mathbb{R}} \left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}dx =& \frac{1}{4}\frac{d}{dt}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^4(\mathbb{R})}^4 + A\beta\varepsilon\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,\\
2\int_{\mathbb{R}} \left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}dx =& -2A\beta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx,\\
-\beta\int_{\mathbb{R}} \left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}\right)\partial_{txx}^3 u_{\varepsilon,\beta}dx =& 3\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{tx}^2 u_{\varepsilon,\beta}dx + A\beta^2\varepsilon\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
\beta^2\int_{\mathbb{R}} \left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}\right)\partial_{txxxx}^5 u_{\varepsilon,\beta}dx =& -3\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{txxx}^4 u_{\varepsilon,\beta}dx\\
&+ A\beta^3\varepsilon\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L(\mathbb{R})}^2,\\
\varepsilon\int_{\mathbb{R}} \left(u_{\varepsilon,\beta}^3 - A\beta\varepsilon\partial_{txx}^3 u_{\varepsilon,\beta}\right)\partial_{xx}^2 u_{\varepsilon,\beta}dx =& -3\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&- \frac{A\beta\varepsilon^2}{2}\frac{d}{dt}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

integrating (3.8) on $\mathbb{R}$, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^4(\mathbb{R})}^4 + \frac{A\beta\varepsilon^2}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\right)\\
&\quad + A\beta\varepsilon\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + A\beta^2\varepsilon\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad + A\beta^3\varepsilon\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + 3\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&= 2A\beta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{txx}^3 u_{\varepsilon,\beta}dx - 3\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{tx}^2 u_{\varepsilon,\beta}dx\\
&\quad + 3\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{txxx}^4 u_{\varepsilon,\beta}dx.
\end{aligned} \tag{3.9}$$

It follows from (2.25), (2.27), (2.31), (2.32) and (3.9) that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^4(\mathbb{R})}^4 + \frac{A\beta\varepsilon^2}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\right)\\
&\quad + \frac{A\beta\varepsilon}{2}\left\|\partial_{tx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{A\beta^2\varepsilon}{2}\left\|\partial_{txx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad + \varepsilon\left(3 - \frac{2C_0 D}{A} - 2A\right)\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad + \frac{A\beta^3\varepsilon}{2}\left\|\partial_{txxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \le 0.
\end{aligned}$$

where D is a positive constant which will be specified later.
We search a constant A such that

$$3 - \frac{2C_0 D}{A} - 2A > 0,$$

that is

$$2A^2 - 3A + 2C_0 D < 0. \tag{3.10}$$

A does exist if and only if

$$9 - 16C_0 D > 0. \tag{3.11}$$

Choosing

$$D = \frac{1}{16C_0}, \tag{3.12}$$

it follows from (3.10) and (3.11) that there exist $0 < A_1 < A_2$, such that for every

$$A_1 < A < A_2, \tag{3.13}$$

(3.10) holds. Hence, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{A\beta\varepsilon^2}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + \frac{A\beta\varepsilon}{2}\left\|\partial^2_{tx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{A\beta^2\varepsilon}{2}\left\|\partial^3_{txx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \varepsilon K_1\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{A\beta^3\varepsilon}{2}\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le 0.
\end{aligned} \tag{3.14}$$

where K_1 is a fixed positive constant. Integrating (3.14) on $(0,t)$, from (3.4), we have

$$\begin{aligned}
&\frac{1}{4}\|u_{\varepsilon,\beta}(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{A_3\beta\varepsilon^2}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \frac{A_3\beta\varepsilon}{2}\int_0^t\left\|\partial^2_{tx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds + \frac{A_3\beta^2\varepsilon}{2}\int_0^t\left\|\partial^3_{txx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad + \varepsilon K_1\int_0^t\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \frac{A_3\beta^3\varepsilon}{2}\int_0^t\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le C_0,
\end{aligned}$$

Hence,

$$\begin{aligned}
\|u_{\varepsilon,\beta}(t,\cdot)\|_{L^4(\mathbb{R})} \le& C_0,\\
\sqrt{\beta}\varepsilon\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})} \le& C_0,\\
\beta\varepsilon\int_0^t\left\|\partial^2_{tx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le& C_0,\\
\beta^2\varepsilon\int_0^t\left\|\partial^3_{txx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le& C_0,\\
\varepsilon\int_0^t\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}ds \le& C_0,\\
\beta^3\varepsilon\int_0^t\left\|\partial^4_{txxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le& C_0,
\end{aligned}$$

for every $t > 0$. □

Now, we are ready for the proof of Theorem 3.1.

Proof of Theorem 3.1. Let us consider a compactly supported entropy–entropy flux pair (η, q). Multiplying (3.3) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}
\partial_t\eta(u_{\varepsilon,\beta}) + \partial_x q(u_{\varepsilon,\beta}) =& \varepsilon\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta} - \beta\eta'(u_{\varepsilon,\beta})\partial^3_{txx}u_{\varepsilon,\beta} + \beta^2\eta'(u_{\varepsilon,\beta})\partial^5_{txxxx}u_{\varepsilon,\beta}\\
=& I_{1,\varepsilon,\beta} + I_{2,\varepsilon,\beta} + I_{3,\varepsilon,\beta} + I_{4,\varepsilon,\beta} + I_{5,\varepsilon,\beta} + I_{6,\varepsilon,\beta},
\end{aligned}$$

where

$$
\begin{aligned}
I_{1,\varepsilon,\beta} &= \partial_x(\varepsilon\eta'(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}),\\
I_{2,\varepsilon,\beta} &= -\varepsilon\eta''(u_{\varepsilon,\beta})(\partial_x u_{\varepsilon,\beta})^2,\\
I_{3,\varepsilon,\beta} &= \partial_x(-\beta\eta'(u_{\varepsilon,\beta})\partial^2_{tx} u_{\varepsilon,\beta}),\\
I_{4,\varepsilon,\beta} &= \beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^2_{tx} u_{\varepsilon,\beta},\\
I_{5,\varepsilon,\beta} &= \partial_x(\beta^2\eta'(u_{\varepsilon,\beta})\partial^4_{txxx} u_{\varepsilon,\beta}),\\
I_{6,\varepsilon,\beta} &= -\beta^2\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^4_{txxx} u_{\varepsilon,\beta}.
\end{aligned}
\tag{3.15}
$$

Following Theorem 2.1, we have that $I_{1,\varepsilon,\beta}$, $I_{3,\varepsilon,\beta}$, $I_{5,\varepsilon,\beta} \to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $\{I_{2,\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$, $I_{4,\varepsilon,\beta}$, $I_{6,\varepsilon,\beta} \to 0$ in $L^1((0,T)\times\mathbb{R})$.

Arguing as Theorem 2.1, we get (3.6) and (3.7). □

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